Semigroups on pairs of Banach lattices

Christian Seifert and Marcus Waurick

Abstract

We give a characterization of a certain estimate relating two positive semigroups on general Banach lattices to one another in terms of corresponding estimates for the respective generators and of estimates for the respective resolvents. The results have applications to kernel estimates for semigroups induced by accretive and non-local forms on σ-finite measure spaces.

Keywords: positive \( C_0 \)-semigroups, Banach lattices, kernel estimates, perturbed semigroups.

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1 Introduction

In this note, we shall further investigate a result recently obtained in [9]. That is to say that for Banach lattices \( X, Y, Z \) and positive \( C_0 \)-semigroups \( S \) and \( T \) appropriately acting on \( X, Y \) and \( Z \), we give a characterization of the following variation of constants type estimate

\[
\langle T(t)u, v' \rangle_{X \times X'} \leq \langle S(t)u, v' \rangle_{X \times X'} + C \int_0^t \langle T(t-s)u, g \rangle_{Z \times Z'} \langle f, S(s)'v' \rangle_{Y \times Y'} \, ds \quad (t \geq 0)
\]

for suitable positive elements \( u, f, g, v' \). We characterize this estimate in terms of corresponding estimates involving the generators of \( S \) and \( T \) and their resolvents. We shall also provide a strong formulation of this characterization, which can be viewed as a perturbation result for positive semigroups on general Banach lattices. The results have applications to kernel estimates for positive semigroups acting on \( L_p \)-spaces. This leads to a further generalization of the heat kernel estimates obtained in [4]. In fact, consider a closed accretive form \( \tau_0 \) (see below for a definition) on \( L_2(\Omega, m) \) for some measure space \( (\Omega, m) \) inducing a positive and \( L_\infty \)-contractive (i.e. submarkovian) semigroup \( S \) with a specific property on the domain \( D(\tau_0) \). Perturbing \( \tau_0 \) additively by \( \tau_j \) given by

\[
\tau_j(u) := \int_{\Omega \times \Omega} (u(x) - u(y))^2 j(x, y)dm^2(x, y)
\]

for some measurable (possibly non-symmetric and unbounded) \( j : \Omega \times \Omega \to [0, \infty) \), we obtain kernel estimates for the semigroup induced by \( \tau := \tau_0 + \tau_j \) in terms of the kernel of \( S \) for \( j \).
Thus, it suffices to show that for bounded \( j \) and regular Dirichlet forms \( \tau_0 \).

This note consists of 4 parts. In Section 2, we shall give a technical lemma (Lemma 2.1) to be used in the main theorem of this article (Theorem 3.6). As it turned out, this lemma can be viewed to describe a behavior of certain convolutions and, thus, may be of independent interest. Section 3 deals with the framework and the statement of our main lemma. The next section states variants of the main theorem and the concluding section deals with an application to heat kernel estimates for non-local forms.

## 2 An auxiliary result

**Lemma 2.1.** For \( n \in \mathbb{N} \) let \( f_n, g_n : [0, \infty) \times \mathbb{N}_0 \to \mathbb{R} \) be continuous in the first variable, \( f, g : [0, \infty) \to \mathbb{R} \), with \( f_n(\cdot, n) \to f \) uniformly on compacts, \( g_n(\cdot, n) \to g \) uniformly on compacts. Assume \( f_n(t, l) = f(l \cdot \frac{1}{n}, l) \) for all \( n \in \mathbb{N}, l \in \mathbb{N}, t \geq 0 \), and similarly for \( g_n \). Then

\[
\frac{1}{n-1} \sum_{l=1}^{n-1} f_{n-1}(t, n-l) g_{n-1}(t, l) \to \int_0^1 f(t(1-s))g(ts) \, ds \quad (t \geq 0).
\]

**Proof.** For \( n \geq 2 \) define \( \mu_n := \frac{1}{n-1} \sum_{l=1}^{n-1} \delta_l, \delta_l \) the Dirac measure at \( l \), and \( \nu_n := \mu_n(n \cdot) \). Then, for \( \lambda \) denoting the Lebesgue measure, \( \nu_n \to \lambda \) weakly on \((0, 1)\). Let \( t \geq 0 \). We compute

\[
\frac{1}{n-1} \sum_{l=1}^{n-1} f_{n-1}(t, n-l) g_{n-1}(t, l) = \int_0^n f_{n-1}(t, n-l) g_{n-1}(t, l) \, d\mu_n(l)
\]

\[
= \int_0^1 f_{n-1}(t, n(1-l)) g_{n-1}(t, nl) \, d\nu_n(l)
\]

\[
= \int_0^1 f_{n(1-l)}(t \frac{n}{n-1}(1-l), n(1-l)) g_{nl}(t \frac{n}{n-1} l, nl) \, d\nu_n(l).
\]

Let \( \varepsilon > 0, K := [0, 2t] \). Then there exists \( 0 < a < b < 1 \) such that

\[
\sup_n \left| \left( \int_0^a + \int_b^1 \right) f_{n(1-l)}(t \frac{n}{n-1}(1-l), n(1-l)) g_{nl}(t \frac{n}{n-1} l, nl) \, d\nu_n(l) \right| \leq \varepsilon.
\]

Thus, it suffices to show that

\[
\int_a^b f_{n(1-l)}(t \frac{n}{n-1}(1-l), n(1-l)) g_{nl}(t \frac{n}{n-1} l, nl) \, d\nu_n(l) \to \int_a^b f(t(1-s))g(ts) \, ds.
\]

There exists \( N \in \mathbb{N} \) such that for all \( n \geq N \) we have

\[
\|f_n(\cdot, n) - f\|_{\infty, K}, \|g_n(\cdot, n) - g\|_{\infty, K} \leq \varepsilon,
\]

\[
\left| \int_0^1 f(t(1-s))g(ts) \, d(\nu_n(s) - \lambda(s)) \right| \leq \varepsilon,
\]
and

\[ \sup_{t \in [0,1]} \left| f\left(t \frac{n}{n-1}(1-l)\right)g\left(t \frac{n}{n-1}l\right) - f(t(1-l))g(tl) \right| \leq \varepsilon \]

since \( f \) and \( g \) are uniformly continuous on \( K \). For \( n \geq N \max\{(1-a)^{-1}, (1-b)^{-1}, a^{-1}, b^{-1}\} \geq N \) we obtain

\[ \left| \int_a^b f_n(1-s)(t \frac{n}{n-1}(1-l), n(1-l))g_n(t \frac{n}{n-1}l, nl) d\nu_n(l) - \int_a^b f(t(1-s))g(tsl) ds \right| \]
\[ \leq \int_a^b \left| f_n(1-s)(t \frac{n}{n-1}(1-l), n(1-l))g_n(t \frac{n}{n-1}l, nl) - f(t(1-l))g(t \frac{n}{n-1}l) \right| d\nu_n(l) \]
\[ + \int_a^b \left| f(t(1-s))g(tsl) - f(t(1-l))g(t \frac{n}{n-1}l) \right| d\nu_n(l) \]
\[ + \left| \int_a^b f(t(1-s))g(tsl) d(\nu_n(s) - \lambda(s)) \right| \]
\[ \leq \int_a^b \left| f_n(1-s)(t \frac{n}{n-1}(1-l), n(1-l)) - f(t \frac{n}{n-1}(1-l)) \right| g_n(t \frac{n}{n-1}l, nl) d\nu_n(l) \]
\[ + \int_a^b \left| f(t \frac{n}{n-1}(1-l)) g_n(t \frac{n}{n-1}l, nl) - g(t \frac{n}{n-1}l) \right| d\nu_n(l) + \varepsilon + \varepsilon \]
\[ \leq \left( \sup_{n \in \mathbb{N}, s \in K} |g_n(s, n)| + \sup_{s \in K} |f(s)| \right) \varepsilon + 2\varepsilon. \]

\[ \square \]

3 The main result for positive semigroups

In the following, we shall come to our main result on positive semigroups. For stating the main theorem we need the following notions.

**Definition.** Let \( X, Y \) be Banach spaces over the same field. We say that \((X, Y)\) is compatible (as Banach spaces), if there exists a Hausdorff topological vector space \( V \) such that \( X, Y \subseteq V \) continuously. If, in addition, \( X, Y \) are vector lattices, we say that \((X, Y)\) is compatible (as lattices), if there exists a Hausdorff topological vector space \( V \) being also a vector lattice such that \( X, Y \subseteq V \) continuously and biperiodically. If, in addition, \( X, Y \) are Banach lattices, we say that \((X, Y)\) is compatible (as Banach lattices), if \((X, Y)\) is compatible as lattices with \( X + Y \) (endowed with the norm \( x + y \mapsto \inf\{\|x_1\|_X + \|y_1\|_Y; \ x + y = x_1 + y_2\}\)) being a Banach lattice.

**Remark 3.1.** Let \( X, Y \) be Banach spaces, \((X, Y)\) compatible as Banach spaces, \( X \cap Y \) dense in \( X \) and \( Y \) and

\[ \langle \cdot, \cdot \rangle_{X \times Y'} = \langle \cdot, \cdot \rangle_{Y \times Y'} \]

on \( X \cap Y \times X' \cap Y' \). Let \( T : X \cap Y \to X \cap Y \) be such that \( T \) extends to continuous linear operators \( T_X \) and \( T_Y \) on \( X \) and \( Y \), respectively. Then \( T'_X = T'_Y \) on \( X' \cap Y' \). Indeed, if
$v' \in X' \cap Y'$ then for $x \in X \cap Y$ we compute

$$\langle x, T'_X v' \rangle_{X \times X'} = \langle T_X x, v' \rangle_{X \times X'} = \langle T_x, v' \rangle_{X \times X'},$$

$$= \langle T_x, v' \rangle_{X \times X'} = \langle T'_Y x, v' \rangle_{Y \times Y'} = \langle x, T'_Y v' \rangle_{Y \times Y'}.$$  

Hence, $\langle \cdot, T'_X v' \rangle_{X \times X'} \in X'$ coincides with $\langle \cdot, T'_Y v' \rangle_{Y \times Y'}$ on $X \cap Y$ which is dense in $X$. Interchanging the roles of $X$ and $Y$ we get $T'_X v' = T'_Y v'$.

**Remark 3.2.** (a) Let $X, Y$ be Banach lattices with $(X, Y)$ compatible as Banach lattices. Then $X + Y$ is, by definition, a Banach lattice. Hence, so is $(X + Y)'$. If, in addition, $X \cap Y$ is dense in both $X$ and $Y$ we get $(X + Y)' = X' \cap Y'$ by [5, Therem 2.7.1]. Hence, $X' \cap Y'$ is a Banach lattice as well.

(b) Let $X, Y$ be Banach lattices with $(X, Y)$ compatible as Banach spaces. Then $X' \cap Y'$ is a Banach lattice with the order $0 \leq v' \in X' \cap Y'$ if and only if $v' \geq 0$ in both $X'$ and $Y'$ and $v' \mapsto \max\{\|v'\|_X, \|v'\|_Y\}$ as lattice norm.

(c) Let $X, Y$ be Banach lattices, $X' \cap Y$ dense in $X$ and $Y$, $X' \cap Y'$ dense in $X'$ and $Y'$, $(X, Y)$ compatible as lattices. Then $(X' \cap Y')'$ induces a natural order on $X + Y$ such that $(X, Y)$ is compatible as Banach lattices with $V := (X' \cap Y')'$ as underlying topological vector space. Indeed, by (b), $X' \cap Y'$ is a Banach lattice with $j_{X'}: X' \to X' \cap Y', x' \mapsto x'$ being an order preserving, continuous embedding (similarly for $Y'$). Hence, the dual $j_{X'}'$ is continuous and order preserving. Next, $j_{X'}'$ is one-to-one on $X$: Let $x \in X$ with $j_{X'}'(x) = 0$. Then, for all $v' \in X' \cap Y'$ we compute $0 = \langle j_{X'}'(x), v' \rangle = \langle x, j_{X'}'(v') \rangle = \langle x, v' \rangle$, implying $x = 0$ by the density of $X' \cap Y'$ in $X'$.

**Definition.** Let $X, Y$ be Banach spaces over the same field, $(X, Y)$ compatible, $T_X$ and $T_Y$ semigroups on $X$ and $Y$, respectively. We say that $T_X$ and $T_Y$ are consistent if $T_X$ and $T_Y$ are $C_0$-semigroups on $X$ and $Y$, respectively, and $T_X(t) u = T_Y(t) u$ for all $t \geq 0$ and $u \in X \cap Y$. We say that $T_X$ is consistent with $(X, Y)$ if there exists a semigroup $T_Y$ on $Y$ such that $T_X$ and $T_Y$ are consistent.

For formulating the theorem, we shall need the following notion from semigroup theory, see e.g. [6, p 61].

**Definition.** Let $X$ be a Banach space, $T$ a $C_0$-semigroup on $X$ with generator $A$. Then $A^\sigma$ given by

$$A^\sigma := \left\{ (x', y') \in X' \times X'; w^*- \lim_{t \downarrow 0} \frac{1}{t} (T(t)' x' - x') = y' \right\},$$

is called the weak*-generator of $T(\cdot)'$.

**Remark 3.3.** It can be shown that $A^\sigma = A'$, the dual operator of the generator $A$ of $T$, see e.g. [6, p 61]. If $X$ is a Banach lattice, then so is $X'$ and if $(\lambda - A)^{-1}$ is positive for some real $\lambda \in \rho(A)$, then so is $(\lambda - A')^{-1} = ((\lambda - A)^{-1})'$. Note that, in general, $A'$ need not be densely defined anymore.

The main theorem and its variants in the next section will be formulated within the following situation:
Hypothesis 3.4. Let \( X, Y, Z \) be real Banach lattices such that \((X, Y)\) and \((X, Z)\) are compatible as Banach spaces, \( X \cap Y \) dense in \( X \) and \( Y \). Let \( S, T \) be positive \( C_0 \)-semigroups on \( X \) with generators \( A_S, A_T \), respectively. Assume that \( S \) is consistent with \((X, Y)\) and \( T \) is consistent with \((X, Z)\), and let

\[
\langle \cdot, \cdot \rangle_{X \times X'} = \langle \cdot, \cdot \rangle_{Y \times Y'}
\]
on \( X \cap Y \times X' \cap Y' \).

Lemma 3.5. Let \( Y, Z \) be Banach spaces, \( S \) a \( C_0 \)-semigroup on \( Y \), \( T \) a \( C_0 \)-semigroup on \( Z \). Let \( f \in Y \), \( v' \in Y' \), \( u \in Z \), \( g' \in Z' \). Then

\[
1 \int_0^t \langle T(t-s)u, g' \rangle_{Z \times Z'} \langle f, (s)\rangle_{Y \times Y'} ds \to \langle u, g' \rangle_{Z \times Z'} \langle f, v' \rangle_{Y \times Y'} \quad (t \to 0).
\]

Proof. We compute

\[
\frac{1}{t} \int_0^t \langle T(t-s)u, g' \rangle_{Z \times Z'} \langle f, (s)g' \rangle_{Y \times Y'} ds - \langle u, g' \rangle_{Z \times Z'} \langle f, v' \rangle_{Y \times Y'}
\]

\[
= \frac{1}{t} \int_0^t \langle (T(t-s)u, g' \rangle_{Z \times Z'} \langle f, (s)g' \rangle_{Y \times Y'} ds
\]

\[
+ \frac{1}{t} \int_0^t \langle (T(t-s)-I)u, g' \rangle_{Z \times Z'} \langle f, v' \rangle_{Y \times Y'} ds.
\]

Since \( T \) is locally uniformly bounded and \( S(\cdot)' \) is weakly* continuous, the first term on the right-hand side tends to zero as \( t \to 0 \). Since \( T \) is strongly continuous on \( Z \) the second term on the right-hand side tends also to zero. Thus, the assertion follows. \( \square \)

Theorem 3.6. Assume Hypothesis 3.4. Let \( 0 \leq f \in Y \), \( 0 \leq g' \in Z' \), \( C \geq 0 \). Then the following are equivalent:

(a) For \( t \geq 0 \) and \( 0 \leq u \in X \cap Z \), \( 0 \leq v' \in X' \cap Y' \) we have

\[
\langle T(t)u, v' \rangle_{X \times X'} \leq \langle S(t)u, v' \rangle_{X \times X'} + C \int_0^t \langle (T(t-s)u, g')_{Z \times Z'}, \langle f, (s)g' \rangle_{Y \times Y'} ds.
\]

(b) For all \( 0 \leq u \in D(A_T) \cap Z \), \( 0 \leq v' \in D(A_S^\gamma) \cap Y' \) we have

\[
\langle A_Tu, v' \rangle_{X \times X'} \leq \langle A_S^\gamma v \rangle_{X \times X'} + C \langle u, g' \rangle_{Z \times Z'} \langle f, v' \rangle_{Y \times Y'}.
\]

(c) For \( \lambda \in \mathbb{R} \) sufficiently large and \( 0 \leq u \in X \cap Z \), \( 0 \leq v' \in X' \cap Y' \) we have

\[
\langle (\lambda - A_T)^{-1}u, v' \rangle_{X \times X'} \leq \langle (\lambda - A_S)^{-1}u, v' \rangle_{X \times X'}
\]

\[
+ C \langle (\lambda - A_T)^{-1}u, g' \rangle_{Z \times Z'} \langle f, (\lambda - A_S^\gamma)^{-1}v' \rangle_{Y \times Y'}.
\]

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Proof. “(a)⇒(b)”: Let $0 \leq u \in D(A_T) \cap Z$, $0 \leq v' \in D(A^g_S) \cap Y'$. For $t > 0$ we have
\[
\left\langle \frac{1}{t} \langle T(t)u - u \rangle , v' \right\rangle_{X \times X'} = \left\langle \frac{1}{t} T(t)u , v' \right\rangle_{X \times X'} - \left\langle \frac{1}{t} u , v' \right\rangle_{X \times X'} \leq \left\langle \frac{1}{t} \langle S(t)u - u \rangle , v' \right\rangle_{X \times X'} + C \frac{1}{t} \int_0^t \langle T(t - s)u , g' \rangle_{Z \times Z'} \langle f , S(s)v' \rangle_{Y \times Y'} ds.
\]
Now, the limit $t \to 0$ yields the assertion by Lemma 3.5.

“(b)⇒(c)”: Let $\lambda \in \mathbb{R}$ be sufficiently large, $0 \leq u \in X \cap Z$, $0 \leq v' \in X' \cap Y'$. Then $0 \leq (\lambda - A_T)^{-1} u =: \tilde{u} \in D(A_T) \cap Z$ and $0 \leq (\lambda - A^g_S)^{-1} v' =: \tilde{v}' \in D(A^g_S) \cap Y'$. Furthermore,
\[
\left\langle (\lambda - A_T)^{-1} u , v' \right\rangle_{X \times X'} = \left\langle A_T \tilde{u} , \tilde{v}' \right\rangle_{X \times X'} - \left\langle \tilde{u} , A^g_S \tilde{v}' \right\rangle_{X \times X'} \leq C \left\langle \tilde{u} , g' \right\rangle_{Z \times Z'} (f , \tilde{v}')_{Y \times Y'}
\]
\[
= C \left\langle (\lambda - A_T)^{-1} u , g' \right\rangle_{Z \times Z'} (f , (\lambda - A^g_S)^{-1} v')_{Y \times Y'}.
\]

“(c)⇒(a)”: Let $0 \leq u \in X \cap Z$, $0 \leq v' \in X' \cap Y'$. Let $\lambda \in \mathbb{R}$ be sufficiently large. By induction on $n \in \mathbb{N}$ we obtain
\[
\left\langle (\lambda - A_T)^{-n} u , v' \right\rangle_{X \times X'} \leq \left\langle (\lambda - A_S)^{-n} u , v' \right\rangle_{X \times X'} + C \sum_{i=1}^n \left\langle (\lambda - A_T)^{(i-1)} u , g' \right\rangle_{Z \times Z'} (f , (\lambda - A^g_S)^{-i} v')_{Y \times Y'}.
\]
Indeed, the case $n = 1$ holds by assumption. For the inductive step from $n \in \mathbb{N}$ to $n + 1$ we observe
\[
\left\langle (\lambda - A_T)^{-n-1} u , v' \right\rangle_{X \times X'} = \left\langle (\lambda - A_T)^{-1} (\lambda - A_T)^{-n} u , v' \right\rangle_{X \times X'} \leq \left\langle (\lambda - A_S)^{-1} (\lambda - A_T)^{-n} u , v' \right\rangle_{X \times X'} \quad \text{+ } C \left\langle (\lambda - A_T)^{-1} (\lambda - A_T)^{-n} u , g' \right\rangle_{Z \times Z'} (f , (\lambda - A^g_S)^{-1} v')_{Y \times Y'}
\]
\[
\leq \left\langle (\lambda - A_T)^{-n} u , (\lambda - A^g_S)^{-1} v' \right\rangle_{X \times X'} \quad \text{+ } C \left\langle (\lambda - A_T)^{-1} (\lambda - A_T)^{-n} u , g' \right\rangle_{Z \times Z'} (f , (\lambda - A^g_S)^{-1} v')_{Y \times Y'} \leq \left\langle (\lambda - A_S)^{-n} u , (\lambda - A^g_S)^{-1} v' \right\rangle_{X \times X'} \quad \text{+ } C \sum_{i=1}^n \left\langle (\lambda - A_T)^{(i-1)} u , g' \right\rangle_{Z \times Z'} (f , (\lambda - A^g_S)^{-i} v')_{Y \times Y'}
\]
\[
+ C \left\langle (\lambda - A_T)^{-1} (\lambda - A_T)^{-n} u , g' \right\rangle_{Z \times Z'} (f , (\lambda - A^g_S)^{-1} v')_{Y \times Y'}.
\]
where we used the Remarks 3.3 and 3.1 for noting that \((\lambda - A_S^{-1})^{-1} v'\) defines the same element either if considered to be in \(X'\) or \(Y'\), respectively.

Let \(t > 0\), and let \(n \in \mathbb{N}\) be sufficiently large. Then

\[
\left\langle \left(1 - \frac{n}{t} A_T\right)^{-n} u, v' \right\rangle_{X \times X'} = \left\langle \left(\frac{n}{t}\right)^n \left(\frac{n}{t} - A_T\right)^{-n} u, v' \right\rangle_{X \times X'} \\
\leq \left\langle \left(\frac{n}{t}\right)^n \left(\frac{n}{t} - A_S\right)^{-n} u, v' \right\rangle_{X \times X'} \\
+ C t \frac{1}{n} \sum_{l=1}^{n} \left(\left(\frac{n}{t}\right)^{n+1-l} \left(\frac{n}{t} - A_T\right)^{l-n-1} u, g' \right)_{Z \times Z'} \left\langle f, \left(\frac{n}{t}\right)^l \left(\frac{n}{t} - A_S^{-1}\right)^{-l} v' \right\rangle_{Y \times Y'} \\
= \left\langle \left(1 - \frac{n}{t} A_S\right)^{-n} u, v' \right\rangle_{X \times X'} \\
+ C t \frac{1}{n} \sum_{l=1}^{n} \left(\left(1 - \frac{n}{t} A_T\right)^{-1}\right)^{n+1-l} u, g' \right)_{Z \times Z'} \left\langle f, \left(1 - \frac{n}{t} A_S^{-1}\right)^{-l} v' \right\rangle_{Y \times Y'}.
\]

Set \(f_n(t, l) := \left\langle \left(1 - \frac{n}{t} A_T\right)^{-1}\right)^{l} u, g' \right)_{Z \times Z'}\) and \(g_n(t, l) := \left\langle f, \left(1 - \frac{n}{t} A_S^{-1}\right)^{-l} v' \right\rangle_{Y \times Y'}\). By Lemma 2.1 and the exponential formula we obtain, as \(n \to \infty\),

\[
\left\langle T(t)u, v' \right\rangle_{X \times X'} \leq \left\langle S(t)u, v' \right\rangle_{X \times X'} + C t \int_0^1 \left\langle T(t(1-s))u, g' \right\rangle_{Z \times Z'} \left\langle f, S(ts)v' \right\rangle_{Y \times Y'} ds \\
= \left\langle S(t)u, v' \right\rangle_{X \times X'} + C \int_0^t \left\langle T(t-s)u, g' \right\rangle_{Z \times Z'} \left\langle f, S(s)v' \right\rangle_{Y \times Y'} ds.
\]

**Remark 3.7.** Note that, in the above proof, we did not really use the Banach lattice structure of the Banach spaces involved in the following sense: Let away the restriction to positive elements in any statement and replace the Banach lattices by general Banach spaces satisfying the respective compatibility conditions. Then the estimates stated are a mere equalities (this follows from the replacement \(v'\) with \(-v'\)). Therefore, the theorem for general Banach spaces is of limited applicability. Hence, we shall stick to the Banach lattice case.

## 4 Variants of the main result

We can prove a similar statement to Theorem 3.6 (with essentially the same proof) by using the exponential growth bound for the semigroup \(T\) on \(Z\) and correspondingly the norm estimate for the resolvent \((\lambda - A_T)^{-1}\) on \(Z\):
Theorem 4.1. Assume Hypothesis 3.4. Let \( M \geq 1, \omega \in \mathbb{R} \) such that \( \|T(t)u\|_Z \leq M e^{\omega t} \|u\|_Z \) for all \( t \geq 0 \) and \( u \in Z \).

Let \( 0 \leq f \in Y \). Then the following are equivalent:

(a) There exists \( C \geq 0 \) such that for all \( t \geq 0 \) and \( 0 \leq u \in X \cap Z, \ 0 \leq v' \in X' \cap Y' \) we have

\[
\langle T(t)u, v' \rangle_{X \times X'} \leq \langle S(t)u, v' \rangle_{X \times X'} + C \int_0^t e^{\omega(t-s)} \|u\|_Z \langle f, S(s)v' \rangle_{Y \times Y'} \ ds.
\]

(b) There exists \( C \geq 0 \) such that for all \( 0 \leq u \in D(A_T) \cap Z, \ 0 \leq v' \in D(A_S^g) \cap Y' \) we have

\[
\langle A_Tu, v' \rangle_{X \times X'} \leq \langle u, A_S^gv' \rangle_{X \times X'} + C \|u\|_Z \langle f, v' \rangle_{Y \times Y'}.
\]

(c) There exists \( C \geq 0 \) such that for all \( \lambda \in \mathbb{R} \) sufficiently large and \( 0 \leq u \in X \cap Z, \ 0 \leq v' \in X' \cap Y' \) we have

\[
\langle (\lambda - A_T)^{-1}u, v' \rangle_{X \times X'} \leq \langle (\lambda - A_S)^{-1}u, v' \rangle_{X \times X'} + \frac{C}{\lambda - \omega} \|u\|_Z \langle f, (\lambda - A_S^g)^{-1}v' \rangle_{Y \times Y'}.
\]

In order to obtain a strong version of the preceding theorem, we will have to get rid of the testing with \( v' \). Therefore, we need to detect orderings of elements via testing with a suitable set of functionals. This is the content of the next (standard) lemma, where we skip the Hahn-Banach type argument.

Lemma 4.2. Let \( X \) be a Banach lattice, \( x \in X \). The following are equivalent:

(a) \( x \geq 0 \).

(b) \( \langle x, x' \rangle \geq 0 \) for all \( 0 \leq x' \in X' \).

As a consequence, we obtain the following statement for compatible Banach lattices:

Corollary 4.3. Let \( (X, Y) \) be a pair of compatible Banach lattices, \( X \cap Y \) dense in \( X \) and \( Y \), \( x \in X, y \in Y \). The following are equivalent:

(a) \( x + y \geq 0 \).

(b) \( \langle x + y, v' \rangle \geq 0 \) for all \( 0 \leq v' \in X' \cap Y' \).

Proof. The part \( (a) \Rightarrow (b) \) is easy. For the converse implication observe that, by Remark 3.2, the dual \( (X + Y)' \) coincides with \( X' \cap Y' \). Thus, the assertion follows from Lemma 4.2. \( \square \)

Corollary 4.4. Assume Hypothesis 3.4 and assume that \( (X, Y) \) is compatible as Banach lattice. Let \( M \geq 1, \omega \in \mathbb{R} \) such that \( \|T(t)u\|_Z \leq M e^{\omega t} \|u\|_Z \) for all \( t \geq 0 \) and \( u \in Z \).

Let \( 0 \leq f \in Y \). Then the following are equivalent:

(a) There exists \( C \geq 0 \) such that for all \( t \geq 0 \) and \( 0 \leq u \in X \cap Z \), we have

\[
T(t)u \leq S(t)u + C \int_0^t e^{\omega(t-s)} \|u\|_Z S(s)f \ ds.
\]
(b) There exists $C \geq 0$ such that for all $0 \leq u \in D(A_T) \cap Z$, $0 \leq v' \in D(A_S^2) \cap Y'$ we have

$$\langle A_T u, v' \rangle_{X \times X'} \leq \langle u, A_S^2 v' \rangle_{X \times X'} + C \|u\|_{Z} \|f, v'\|_{Y \times Y'}.$$ 

(c) There exists $C \geq 0$ such that for all $\lambda \in \mathbb{R}$ sufficiently large and $0 \leq u \in X \cap Z$ we have

$$(\lambda - A_T)^{-1}u \leq (\lambda - A_S)^{-1}u + \frac{C}{\lambda - \omega} \|u\|_{Z}(\lambda - A_S)^{-1}f.$$ 

Proof. Using Theorem 4.1, we realize that it suffices to observe that the statements (a),(b) and (c) from Theorem 4.1 are equivalent to the corresponding ones here. For this, note that there is nothing to show for the statement (b) and that the remaining equivalences follow from Corollary 4.3.

Remark 4.5. In Corollary 4.4, instead of the compatibility of $(X,Y)$ as Banach lattices one may assume that $(X,Y)$ is compatible as vector lattice and that the statement in Corollary 4.3 holds as an assumption, i.e., $x + y \geq 0$ if and only if $(x + y, v') \geq 0$ for all $v' \in X' \cap Y'$. The latter are satisfied if $X,Y$ and $Z$ are (possibly different) $L_p$-spaces with $1 \leq p < \infty$. If one of the spaces $X,Y$ or $Z$ coincides with $L_1$ one needs to assume localizable underlying measure spaces in order that $L'_1 = L_\infty$.

Remark 4.6. An analogous statement was proven in [9] for semigroups on $L_2$-spaces, where the perturbation on the right hand side of (a) was $L_\infty$. However, one needs to realize that one can prove a similar statement assuming that $X$ and $Y$ are dual spaces and $X'$ and $Y'$ are replaced by the respective preduals.

5 An example for $L_p$-spaces

Let $(\Omega, m)$ be a localizable measure space, $1 \leq p, q < \infty$. Let $X := L_2(\Omega, m)$ (with inner product $(\cdot | \cdot)$), $Y := L_p(\Omega, m)$, $Z := L_q(\Omega, m)$, with scalar field $\mathbb{R}$. We can now use our result to prove kernels estimates for $T$ in terms of the kernel for $S$.

Corollary 5.1. Let $(\Omega, m)$ be $\sigma$-finite, $p > 1$, $M \geq 1$, $\omega \in \mathbb{R}$ such that $\|T(t)u\|_{Z} \leq Me^{\omega t}\|u\|_{Z}$ for all $t \geq 0$ and $u \in Z$.

If $0 \leq f \in Y$. Assume $S$ has a kernel $k^S : [0, \infty) \times \Omega^2 \rightarrow \mathbb{R}$, i.e.

$$S(t)u = \int_{\Omega} k^S(t, :, y) u(y) \, dm(y).$$

Assume

$$\langle A_T u, v' \rangle_{X \times X'} \leq \langle u, A_S^2 v' \rangle_{X \times X'} + C_0 \|u\|_{Z} \|f, v'\|_{Y \times Y'}$$

for all $0 \leq u \in D(A_T) \cap Z$, $0 \leq v' \in D(A_S^2) \cap Y'$, and some $C_0 \geq 0$.

Then $T$ has a kernel $k^T : [0, \infty) \times \Omega^2 \rightarrow \mathbb{R}$ and there exists $C \geq 0$ such that

$$k^T(t, x, y) \leq k^S(t, x, y) + C \int_{0}^{t} e^{\omega(t-s)} \int_{\Omega} k^S(s, x, y) f(y) \, dm(y) \, ds$$

for $m^2$-a.a. $(x, y) \in \Omega^2$ and $t \geq 0$. 

Proof. The existence of a kernel for $T$ can be achieved as in the proof of [2, Theorem 5.9]. The kernel estimate then follows from the corresponding estimate for the semigroups in Corollary 4.4(a) and [10, Korollar 2.1.11].

**Remark 5.2.** One can formulate and prove a similar corollary as the preceding one in the spirit of Theorem 3.6.

We give a short introduction to forms on $L_2$-spaces; for more information see e.g. [7, 1].

A bilinear map $\tau : D(\tau) \times D(\tau) \to \mathbb{R}$, where $D(\tau)$ is a subspace of $X$, is called a *form*. We write $\tau(u) := \tau(u, u)$ for the corresponding quadratic form. A form $\tau$ is *densely defined* if $D(\tau)$ is dense in $X$. It is called *accretive* if $\tau(u, u) \geq 0$ for all $u \in D(\tau)$. $\tau$ is called *continuous* if there exists $M \geq 0$ such that

$$|\tau(u, v)| \leq M\|u\|_\tau\|v\|_\tau \quad (u, v \in D(\tau)),$$

where $\|\cdot\|_\tau := (\tau(\cdot, \cdot) + \|\cdot\|_X^2)^{1/2}$ is the *form norm*. We say that $\tau$ is closed if $(D(\tau), \|\cdot\|_\tau)$ is complete.

**Remark 5.3.** Let $\tau$ be densely defined, accretive, continuous and closed. Then we can associate an operator $A$ to $\tau$ via

$$A := \{(u, v) \in X \times X; u \in D(\tau), \tau(u, \varphi) = (v \mid \varphi) \quad (\varphi \in D(\tau))\}.$$

Note that $A$ is an $m$-accretive operator, i.e. $(Au \mid u) \geq 0$ for all $u \in D(A)$ and $R(I + A) = X$. Furthermore, $-A$ generates a contractive $C_0$-semigroup $(e^{-tA})_{t \geq 0}$ on $X$.

**Remark 5.4** ([7, Theorem 2.6]). Let $\tau$ be a densely defined accretive continuous closed form in $L_2(\Omega, m)$ (over $\mathbb{R}$), $A$ the associated operator and $T = (e^{-tA})_{t \geq 0}$ the associated $C_0$-semigroup. The following are equivalent.

(a) $T$ is positive, i.e. $T(t)u \geq 0$ for all $0 \leq u \in L_2(\Omega, m)$, $t \geq 0$.

(b) For $u \in D(\tau)$ we have $u^+ \in D(\tau)$ and $\tau(u^+, u^-) \leq 0$.

**Remark 5.5** ([7, Theorem 2.13]). Let $\tau$ be a densely defined accretive continuous closed form in $L_2(\Omega, m)$ (over $\mathbb{R}$), $A$ the associated operator and $T = (e^{-tA})_{t \geq 0}$ the associated $C_0$-semigroup. The following are equivalent.

(a) $T$ is $L_\infty$-contractive, i.e. $\|T(t)u\|_\infty \leq \|u\|_\infty$ for all $u \in L_2(\Omega, m) \cap L_\infty(\Omega, m)$, $t \geq 0$.

(b) For $u \in D(\tau)$ we have $(1 \wedge |u|) \text{sgn } u \in D(\tau)$ and $\tau(u, (|u| - 1)^+ \text{sgn } u) \geq 0$.

Let $\tau_0$ be a densely defined accretive continuous closed form on $L_2(\Omega, m)$ with associated operator $A_0$ and $C_0$-semigroup $S = (e^{-tA_0})_{t \geq 0}$, such that $S$ is positive and contractive in $L_\infty(\Omega, m)$ (cf. the Remarks 5.4 and 5.5). Then $S$ is consistent with $(L_2(\Omega, m), L_p(\Omega, m))$.

Furthermore, let us assume

$$D_{\text{fin}} := \{u \in D(\tau_0); m([u \neq 0]) < \infty\}$$

is dense in $D(\tau_0)$. 

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Let \( j : \Omega \times \Omega \to \mathbb{R} \) be measurable, \( j \geq 0 \), such that \( \int_B j(x, y) \, dm(x, y) < \infty \) for all Borel sets \( B \subseteq \Omega^2 \) such that \( m^2(B) < \infty \). Consider

\[
D(\tau) := \left\{ u \in D(\tau_0); \int_{\Omega \times \Omega} (u(x) - u(y))^2 j(x, y) \, dm(x, y) < \infty \right\},
\]

\[
\tau(u, v) := \tau_0(u, v) + \int_{\Omega \times \Omega} (u(x) - u(y))(v(x) - v(y)) j(x, y) \, dm(x, y).
\]

**Lemma 5.6.** \( \tau \) is densely defined, accretive, continuous and closed.

**Proof.** To show that \( \tau \) is densely defined it suffices to approximate elements of \( D_{\text{fin}} \) by elements of \( D(\tau) \). Let \( u \in D_{\text{fin}} \). Since \( S \) is positive without loss of generality we may assume that \( u \geq 0 \). Since \( S \) is \( L_\infty \)-contractive, we have \( u_n := u \wedge n \in D(\tau_0) \) for all \( n \in \mathbb{N} \). Since \([u_n \neq 0] = [u \neq 0]\) for all \( n \in \mathbb{N} \) we observe

\[
\int_{\Omega \times \Omega} (u_n(x) - u_n(y))^2 j(x, y) \, dm(x, y) \leq (2n)^2 \int_{\{u \neq 0\} \times \{u \neq 0\}} j(x, y) \, dm(x, y) < \infty \quad (n \in \mathbb{N}),
\]

i.e. \( u_n \in D(\tau) \) for all \( n \in \mathbb{N} \). Since \( u_n \to u \) in \( L_2(\Omega, m) \) the form \( \tau \) is densely defined.

Since \( j \geq 0 \) we have \( \tau(u, u) \geq \tau_0(u, u) \geq 0 \) for all \( u \in D(\tau) \), implying that \( \tau \) is accretive.

Since \( \tau_0 \) is continuous there exists \( M \geq 0 \) such that \( ||\tau_0(u, v)||_{\tau_0} \leq M||u||_{\tau_0}||v||_{\tau_0} \) for all \( u, v \in D(\tau_0) \). Since the bilinear form \((u, v) \mapsto \int_{\Omega \times \Omega} (u(x) - u(y))(v(x) - v(y)) j(x, y) \, dm(x, y)\) is symmetric and nonnegative we obtain \( ||.||_{\tau_0} \leq ||.||_{\tau} \) and by the Cauchy-Schwarz inequality

\[
\int_{\Omega \times \Omega} (u(x) - u(y))(v(x) - v(y)) j(x, y) \, dm(x, y)
\]

\[
\leq \left( \int_{\Omega \times \Omega} (u(x) - u(y))^2 j(x, y) \, dm^2(x, y) \right)^{1/2} \left( \int_{\Omega \times \Omega} (v(x) - v(y))^2 j(x, y) \, dm^2(x, y) \right)^{1/2}
\]

\[
\leq ||u||_{\tau} ||v||_{\tau} \quad (u, v \in D(\tau)).
\]

Thus,

\[
||\tau(u, v)|| \leq M||u||_{\tau} ||v||_{\tau} + ||u||_{\tau} ||v||_{\tau} = (M + 1)||u||_{\tau} ||v||_{\tau} \quad (u, v \in D(\tau)),
\]

i.e. \( \tau \) is continuous.

To show closedness of \( \tau \) let \((u_n)_n\) be a \( ||.||_{\tau} \)-Cauchy sequence in \( D(\tau) \) such that \( u_n \to u \) in \( L_2(\Omega, m) \). Without loss of generality (by choosing a suitable subsequence) we may assume that \( u_n \to u \) \( m \)-a.e. Since \( \tau_0 \leq \tau \) the sequence \((u_n)_n\) is also a \( ||.||_{\tau_0} \)-Cauchy sequence. Since \( \tau_0 \) is closed we obtain \( u \in D(\tau_0) \) and \( ||u_n - u||_{\tau_0} \to 0 \). By Fatou’s lemma we have

\[
\int \left( (u_n - u_m)(x) - (u_n - u_m)(y) \right)^2 j(x, y) \, dm^2(x, y)
\]

\[
= \int \liminf_{m \to \infty} \left( (u_m - u_n)(x) - (u_m - u_n)(y) \right)^2 j(x, y) \, dm^2(x, y)
\]

\[
\leq \liminf_{m \to \infty} \int \left( (u_m - u_n)(x) - (u_m - u_n)(y) \right)^2 j(x, y) \, dm^2(x, y)
\]

\[
\leq \liminf_{m \to \infty} ||u_m - u_n||_{\tau}^2 \to 0 \quad (n \to \infty).
\]
In particular, \( u - u_n \in D(\tau) \) and therefore \( u \in D(\tau) \). Furthermore,

\[
\tau(u - u_n) \leq \tau_0(u - u_n) + \liminf_{m \to \infty} \int \left( (u_m - u_n)(x) - (u_m - u_n)(y) \right)^2 j(x, y) \, dm(x, y) \to 0
\]

which implies \( \|u - u_n\|_{\tau} \to 0 \), i.e. \( \tau \) is closed.

Let \( A \) be the operator associated with \( \tau \) and \( T = (e^{-tA})_{t \geq 0} \) be the \( C_0 \)-semigroup.

**Lemma 5.7.** \( T \) is positive and contractive in \( L_\infty(\Omega, m) \), and therefore consistent with \((L_2(\Omega, m), L_q(\Omega, m))\).

**Proof.** Let \( u \in D(\tau) \). Since \( S \) is positive we have \( u^+ \in D(\tau_0) \) and \( \tau_0(u^+, u^-) \leq 0 \). Note that

\[
(u^+(x) - u^+(y))^2 \leq (u(x) - u(y))^2 \quad \text{for } m^2\text{-a.a. } (x, y) \in \Omega^2.
\]

This implies \( u^+ \in D(\tau) \) and

\[
\tau(u^+, u^-) = \tau_0(u^+, u^-) + \int (u^+(x) - u^+(y))(u^-(x) - u^-(y)) \, j(x, y) \, dm(x, y) \leq 0.
\]

Hence, \( T \) is positive.

Since \( S \) is contractive in \( L_\infty(\Omega, m) \) we have \((1 \wedge |u|) \) sgn \( u \in D(\tau_0) \) and \( \tau_0(u, (|u| - 1)^+ \text{ sgn } u) \geq 0 \). Note that for \( v := (1 \wedge |u|) \) sgn \( u \) we have \((v(x) - v(y))^2 \leq (u(x) - u(y))^2 \) for \( m^2\)-a.a. \( (x, y) \in \Omega^2 \). Thus, \( u \in D(\tau) \), and

\[
\tau(u, (|u| - 1)^+ \text{ sgn } u)
= \tau_0(u, (|u|-1)^+ \text{ sgn } u)
+ \int (u(x) - u(y))((|u(x)|-1)^+ \text{ sgn } u(x) - (|u(y)|-1)^+ \text{ sgn } u(y)) \, j(x, y) \, dm(x, y)
\geq 0.
\]

Hence, \( T \) is \( L_\infty \)-contractive. Since \( T \) is \( L_\infty(\Omega, m) \)-contractive and also \( L_2(\Omega, m) \)-contractive, it acts on the whole scale of \( L_p(\Omega, m) \) with \( 2 \leq p < \infty \).

We distinguish the following cases:

**Case 5.8** \((1 \leq p, q < \infty)\). Assume

\[
\int j(x, \cdot) \, dm(x), \int j(\cdot, y) \, dm(y) \in L_\infty(\Omega, m).
\]

**Case 5.9** \((1 \leq p \leq 2, 1 \leq q < \infty)\). Assume

\[
\int j(x, \cdot) \, dm(x), \int j(\cdot, y) \, dm(y) \in L_\infty(\Omega, m) + L_{\frac{2p}{2-p}}(\Omega, m).
\]

**Case 5.10** \((1 \leq p < \infty, 2 \leq q < \infty)\). Assume

\[
\int j(x, \cdot) \, dm(x), \int j(\cdot, y) \, dm(y) \in L_\infty(\Omega, m) + L_{\frac{2q}{q-2}}(\Omega, m).
\]
\textbf{Case 5.11} \((1 \leq p \leq 2, \ 2 \leq q < \infty)\). Assume
\[
\int j(x, \cdot) \, dm(x) \leq L_{p} (\Omega, m) + L_{\frac{2p}{2-p}} (\Omega, m) + L_{\frac{2q}{q-2}} (\Omega, m) + L_{\frac{2q}{q-p}} (\Omega, m).
\]

Assuming one of the above cases is satisfied, we further assume that \(x \mapsto \|j(x, \cdot)\|_{q'}\), \(y \mapsto \|j(\cdot, y)\|_{q'}\) \(\in L_{p}(\Omega, m)\). For \(0 \leq u \in D(A) \cap L_{q}(\Omega, m)\), \(0 \leq v \in D(A^\sigma_0) \cap L_{q'}(\Omega, m)\), we compute
\[
(-Au \cdot v) - (u \cdot A^\sigma_0 v) = \tau_0(u, v) - \tau(u, v)
\]
\[
= - \int_{\Omega \times \Omega} (u(x) - u(y))(v(x) - v(y)) j(x, y) \, dm(x, y)
\]
\[
\leq \int_{\Omega \times \Omega} u(y)v(x) j(x, y) \, dm(x, y) + \int_{\Omega \times \Omega} u(x)v(y) j(x, y) \, dm(x, y)
\]
\[
\leq \int_{\Omega} v(x) \int_{\Omega} u(y) j(x, y) \, dm(y) \, dm(x)
\]
\[
+ \int_{\Omega} v(y) \int_{\Omega} u(x) j(x, y) \, dm(y) \, dm(x)
\]
\[
\leq \int_{\Omega} v(x) \|u\|_{q} \|j(x, \cdot)\|_{q'} \, dm(x) + \int_{\Omega} v(y) \|u\|_{q} \|j(\cdot, y)\|_{q'} \, dm(y)
\]
\[
= \|u\|_{q} \left\langle [x \mapsto \|j(x, \cdot)\|_{q'}] + [y \mapsto \|j(\cdot, y)\|_{q'}], v \right\rangle_{p,p'}.
\]

Thus, we can apply Corollary 4.4 (at least for \(p > 1\)) to obtain an estimate of the semigroup \(T\) in terms of the semigroup \(S\), and, similarly, we obtain estimates for the respective integral kernels assuming \((\Omega, m)\) is \(\sigma\)-finite, \(S\) has a kernel and applying Corollary 5.1.

\textbf{References}


Christian Seifert  
Technische Universität Hamburg-Harburg  
Institut für Mathematik  
21073 Hamburg, Germany  
christian.seifert@tuhh.de

Marcus Waurick  
Technische Universität Dresden  
Fachrichtung Mathematik  
01062 Dresden, Germany  
marcus.waurick@tu-dresden.de